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Global stability of a class of delay differential systems[☆]

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ABSTRACT

This paper is concerned with a class of systems of delay differential equations which are defined on the nonnegative function space. Under proper conditions, we employ a novel proof to establish several criteria of the global stability of positive equilibrium. Moreover, we give two examples to illustrate our main results.

1. Introduction

In this paper, we consider the following systems of delay differential equations:

$$x'(t) = f(x_t) \tag{1.1}$$

where $f : \prod_{i=1}^{n} C([-r_i, 0], R^1_+) \to R^n$ is a locally Lipschitz map, $f(\widehat{0}) = f(\widehat{x^*}) = 0$ with $x^* \stackrel{\Delta}{=} (x^*_1, x^*_2, \dots, x^*_n) \in Int\, R^n_+$, and r_i is a positive constant with $i \in \{1, 2, \dots, n\}$. When $n = 1$, Eq. (1.1) is a class of general scalar delay differential equations, including Nicholson's blowflies equation and the logistic type delay equation. Recently, Nicholson's blowflies equation and the logistic type delay equation serve as the models for some population dynamics and ecology problems which have been extensively studied by many authors, and many interesting results have been obtained, see for example [1–9]. Here we shall investigate the global stability of positive equilibrium of (1.1). For the system with a large n, there have been several methods such as Lyapunov functional method, invariance principles of Lyapunov–Rarumikhin type and monotone method which may be applied to investigate the global dynamics of (1.1) (see [1,10–15,2,16,3–9]). Before illustrating our study methods, we shall recall one recent paper [16] on the global attractivity of the positive steady state of the diffusive Nicholson's equation. By combining a dynamical systems argument with maximum principle and some subtle inequalities, the authors of [16] have employed such a synthetic method to obtain the global attractivity of the positive steady state of the diffusive Nicholson's equation with homogeneous Neumann boundary value under a condition that makes the equation a nonmonotone dynamical system. This synthetic method motivates that we integrated Lyapunov functional with other methods to analyse the global stability of positive equilibrium of (1.1).

In fact, we first construct a "proper" Lyapunov functional under some appropriate conditions. However, we usually cannot deduce Lyapunov functional's derivatives along (1.1). Therefore, some alternative methods are required. Then, by the

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contrary argument and comparison technique, we show that Lyapunov functional along every nontrivial solution of (1.1) is non-increasing, and also either strictly decreasing or eventually 0. Finally, by combining the basic theory of Lyapunov direct method with the properties of Lyapunov functional and the dynamical properties, we obtain the global stability of (1.1).

As some applications, we also consider the following Nicholson's blowflies equation with patch structure:

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + \beta x_i(t-r_i)e^{-x_i(t-r_i)} - dx_i(t), \quad i = 1, 2, \ldots, n, \tag{1.2}$$

and delayed logistic equation with patch structure:

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + k_i x_i(t)\left[\alpha_i - b_{i0}x_i(t) - \sum_{j=1}^{m} b_{ij}x_i(t-r_j)\right], \quad i = 1, 2, \ldots, n. \tag{1.3}$$

The remaining part of this paper is organized as follows. In Section 2 we give some basic notations. By applying Lyapunov direct method, we also propose several criteria of the global stability of (1.1). In Section 3, some applications are made to Nicholson's blowflies equation and/or delayed logistic equation with patch structure.

2. Main results

In this paper, we denote by $R^n(R^n_+)$ the set of all (nonnegative) real vectors. Let $(r_1, r_2, \ldots, r_n) \in IntR^n_+$ be given and $C = \prod_{i=1}^{n} C([-r_i, 0], R^1)$ be a Banach space equipped with the usual supremum norm $\|\cdot\|$, and let $C_+ = \prod_{i=1}^{n} C([-r_i, 0], R^1_+)$, $N = \{1, 2, \cdots, n\}$ and $r = \max_{i\in N} r_i$. If $x_i(t)$ is defined on $[-r_i, \sigma)$, for any $\sigma > 0$ and $i \in N$, then we define $x_t \in C$ as $x_t = (x_t^1, x_t^2, \ldots, x_t^n)$ where $x_t^i(\theta) = x_i(t+\theta)$ for all $\theta \in [-r_i, 0]$ and $i \in N$. For $x = (x_1, x_2, \ldots, x_n) \in R^n$, we write $\widehat{x}$ for the element of C satisfying $\widehat{x} = (\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_n})$ where $(\widehat{x_i})(\theta_i) = x_i$ for all $\theta_i \in [-r_i, 0]$ and $i \in N$.

We write $x_t(\varphi)(x(t, \varphi))$ for a solution of the initial value problem (1.1) with $x_0 = \varphi$. Also, let $[0, \eta(\varphi))$ be the maximal right-interval of the existence of $x_t(\varphi)$.

We first introduce the following proposition:

Proposition 2.1. *Suppose that the system (1.1) has a unique and bounded solution $x_t(\varphi)$ on $\mathbf{R}^1_+$ with $x_0 = \varphi \in C_+$. Let $V(\varphi) = \|\varphi - x^*\| \overset{\Delta}{=} \sup\{|\varphi_i(\theta) - x_i^*| : i \in N$ and $\theta \in [-r_i, 0]\}$ for all $\varphi \in C_+$. Moreover, there exist $T_1 \geq 0$ and $M \subseteq \{\varphi \in C_+ : \varphi(0) = 0\}$ with $0 \in M$ such that the following conditions hold:*

(i) *$x_t(\varphi) \in Int(C_+)$ for $\varphi \in C_+ \setminus M$ and $t \geq T_1$.*
(ii) *If $\varphi \in C_+ \setminus M$ and $x_t(\varphi) \not\equiv x^*$ for all $t \in R^1_+$, then $V(x_t(\varphi))$ is strictly decreasing for $t \geq T_1 \geq 0$.*
(iii) *$\varphi \in C_+ \setminus M$ and $V(x_t(\varphi))$ is non-increasing on for $t \geq T_1$.*

Then x^* is a globally stable equilibrium on $C_+ \setminus M$.

Proof. We now prove that x^* is a stable equilibrium. For any $\epsilon > 0$, it follows from the continuous dependence of the solution of (1.1) that there is $\delta \in [0, \min\{\epsilon, \min\{x_i^* : i \in N\}\})$ such that $V(x_t(\varphi)) < \epsilon$ for all $t \in [0, T_1]$ and φ with $V(\varphi) < \delta$. On the other hand, assumption (iii) implies $V(x_t(\varphi)) \leq V(x_{T_1}(\varphi)) < \epsilon$ for all $t \in [T_1, +\infty)$ and φ with $V(\varphi) < \delta$. Hence, $V(x_t(\varphi)) < \epsilon$ for all $t \in R_+$ and φ with $V(\varphi) < \delta$. So, x^* is a stable equilibrium.

We next show that x^* attracts $C_+ \setminus M$. Give $\varphi \in C_+ \setminus M$. From (iii), we have that $\lim_{t\to+\infty} V(x_t(\varphi)) = \alpha$ for some $\alpha \geq 0$. We shall show $\alpha = 0$; otherwise, $\alpha > 0$. Here we denote ω-limit set by $\omega(\varphi)$. Then $\omega(\varphi)$ is nonempty, compact and invariant and $V(\psi) = \alpha$ for all $\psi \in \omega(\varphi)$. Note that assumptions (i) and (iii) imply $\omega(\varphi) \neq \{0\}$. Hence by the invariance of $\omega(\varphi)$, we have $\omega(\varphi) \setminus M \neq \emptyset$. So we may choose $\psi^* \in \omega(\varphi) \setminus (M \cup \{x^*\})$. It follows from the assumption (ii) that $V(x_t(\psi^*))$ is strictly decreasing on $[T_1, +\infty)$. But, in view of the invariance of $\omega(\varphi)$ and the choice of α, we have $V(x_t(\psi^*)) \equiv \alpha$ for all $t \in R_+$, a contradiction. Thus, $\lim_{t\to+\infty} V(x_t(\varphi)) = \lim_{t\to+\infty} \|x_t(\varphi) - x^*\| = 0$. This completes the proof of Proposition 2.1. □

In this paper, we shall introduce the following assumptions to guarantee the global stability of system (1.1):

(A1) If $\varphi \in C_+$ and $t \in [0, \eta(\varphi))$, then $x_t(\varphi) \in C_+$;
(A2) If $\varphi \in C_+ \setminus \{0\}$ and $t \in (r, \eta(\varphi))$, then $x_t(\varphi) \in IntC_+$;
(A3) $f(\widehat{0}) = f(\widehat{x^*}) = 0$ where $x^* \overset{\Delta}{=} (x_1^*, x_2^*, \ldots, x_n^*) \in IntR^n_+$;
(A4) For some i and $\varphi \in C_+ \setminus \{x^*\}$ such that $\varphi_i(0) - x_i^* \geq \|\varphi - x^*\|$, we have $f_i(\varphi) < 0$;
(A5) For some i and $\varphi \in IntC_+ \setminus \{x^*\}$ such that $x_i^* - \varphi_i(0) \geq \|\varphi - x^*\|$, we have $f_i(\varphi) > 0$.

Proposition 2.2. *Let* (A1) *and* (A4) *hold. If $\varphi \in C_+ \setminus \{0\}$, then the set of $\{x_t(\varphi) : t \in [0, \eta(\varphi))\}$ is bounded and hence $\eta(\varphi) = +\infty$.*

Proof. Let $\varphi \in C_+ \setminus \{0\}$. We claim that $\|x_t(\varphi) - x^*\| \leq \|\varphi\| + 2\|x^*\|$ for all $t \in [0, \eta(\varphi))$. Otherwise, there exists $t_1 \in (0, \eta(\varphi))$ such that

$$\|x_{t_1}(\varphi) - x^*\| > \|\varphi\| + 2\|x^*\|.$$

Let

$$t^* = \inf\{t : t \in [0, \eta(\varphi)), \|x_t(\varphi) - x^*\| > \|\varphi\| + 2\|x^*\|\}.$$

Then, $t^* \in (0, t_1)$, and

$$\|\varphi\| + 2\|x^*\| = \|x_{t^*}(\varphi) - x^*\| \geq \|x_t(\varphi) - x^*\| \quad \text{for all } t \in [0, t^*]. \tag{2.1}$$

In view of (2.1) and $x_t(\varphi) \in C_+$, there exist $t^{**} \in [0, t^*]$ and $i \in N$ such that

$$x_i(t^{**}, \varphi) - x_i^* = |x_i(t^{**}, \varphi) - x_i^*| = \|\varphi\| + 2\|x^*\| \geq \|x_{t^{**}}(\varphi) - x^*\|.$$

By (A_4), we have $f_i(x_{t^{**}}(\varphi)) < 0$. On the other hand, it follows from the choice of t^{**} and system (1.1) that $f_i(x_{t^{**}}(\varphi)) = x_i'(t^{**}, \varphi) \geq 0$, a contradiction. This implies that the claim holds. Thus,

$$\|x_t(\varphi)\| \leq \|x_t(\varphi) - x^*\| + \|x^*\| \leq \|\varphi\| + 3\|x^*\|, \quad \text{for all } t \in [0, \eta(\varphi)).$$

Finally, according to Theorem 3.1 in [17], we easily obtain $\eta(\varphi) = +\infty$. □

Theorem 2.1. *Let* (A1)–(A5) *hold. Then x^* is a globally stable equilibrium on $C_+ \setminus \{0\}$.*

Proof. Define $V : C_+ \to C_+$ such that $V(\varphi) = \|\varphi - x^*\| \triangleq \sup\{\|\varphi_i(\theta) - x_i^*\| : i \in N \text{ and } \theta \in [-r_i, 0]\}$. Suppose $\varphi \in C_+ \setminus \{0\}$. Obviously, by (A2), we have

$$x_t(\varphi) \in IntC_+ \quad \text{for all } t > r.$$

We next show that the following claims are true:

Claim (i) If there exists $T_0 \geq 0$ such that $x_{T_0}(\varphi) = x^*$, then $x_t(\varphi) = x^*$ for all $t \geq T_0$.

Claim (ii) If $x_t(\varphi) \neq x^*$ for all $t \in R_+^1$, then $V(x_t(\varphi))$ is strictly decreasing on $(r, +\infty)$.

Claim (iii) $V(x_t(\varphi))$ is non-increasing on $(r, +\infty)$.

To complete the proof, we distinguish three cases.

Case (i) Obviously, Claim (i) follows from (A3).

Case (ii) If Claim (ii) does not hold, there exist $t_1, t_2 \in (r, +\infty)$ such that $t_1 < t_2$ and $V(x_{t_2}(\varphi)) \geq V(x_{t_1}(\varphi))$. By the choice of t_1 and t_2, we can deduce that there exists $t_3 \in [t_1, t_2]$ such that $\|x(t_3, \varphi) - x^*\| = V(x_{t_3}(\varphi))$. Thus, there exists $i \in N$ such that either $x_i(t_3, \varphi) - x_i^* = V(x_{t_3}(\varphi))$ or $x_i(t_3, \varphi) - x_i^* = -V(x_{t_3}(\varphi))$. Note that $x_{t_3}(\varphi) \in Int(C_+) \setminus \{x^*\}$. If the former holds, then by (A4) we have $f_i(x_{t_3}(\varphi)) < 0$. On the other hand, it follows from (1.1) that $f_i(x_{t_3}(\varphi)) = x_i'(t_3, \varphi) \geq 0$, a contradiction. If the latter holds, then by (A5), (1.1) and a similar discussion as above, we can deduce a contradiction. Therefore, Claim (ii) follows.

Case (iii) Obviously, Claims (i) and (ii) imply Claim (iii).

Therefore, Theorem 2.1 follows from Proposition 2.1. This completes the proof of Theorem 2.1. □

The above Theorem 2.1 is not completely satisfactory since it does not characterize the global stability of (1.1) solely in terms of properties of f. In the following, we give sufficient conditions on f to guarantee (A1) and (A2).

(A11) There exist $k = (k_1, k_2, \ldots, k_n) \in R_+^n$ and a nonnegative map $g : C_+ \to R_+^n$ such that for all $\varphi \in C_+$, $f(\varphi) = -diag(k_1, k_2, \ldots, k_n)\varphi(0) + g(\varphi)$.

(A12) For each $\varphi \in C_+$, denote $I = \{i : \varphi_i(\theta_i) = 0 \text{ for all } \theta_i \in [-r_i, 0]\}$ and $J = \{i : \varphi_i(\theta_i) > 0 \text{ for all } \theta_i \in [-r_i, 0]\}$. If $I \neq \emptyset$, $J \neq \emptyset$ and $I^\sharp + J^\sharp = n$, then there exists $i \in I$ such that $f_i(\varphi) > 0$.

Lemma 2.1. *Let* (A11), (A12) *and* (A4) *hold. Then for any $\varphi \in C_+$, we have either $x_t(\varphi) = 0$ for all $t \geq 0$ or $x_t(\varphi) \in IntC_+$ for all $t \geq (n + 2)r$.*

Proof. Let $\varphi \in C_+$. Using assumption (A11) and Theorem 5.2.1 in [1, p. 81], we have $x_t(\varphi) \in C_+$ for all $t \in [0, \eta(\varphi))$, which, together with (A4), implies that the conclusion of Proposition 2.2 holds. Moreover, if $\varphi_i(0) > 0$ for some i, then $x_i(t, \varphi) > 0$ for all $t \in R_+$. Hence for any i, either $x_i(t, \varphi) > 0$ for all $t \geq r$ or $x_i(t, \varphi) = 0$ for all $t \in [0, r]$. Assume, by way of contradiction, that conclusion of Lemma 2.1 does not hold. Then, there exists $t_1 \in [0, r]$ such that $x_i(t_1, \varphi) > 0$ for some i.

Let $M_t = \{i \in N : x_i(t, \varphi) > 0\}$, where $t \geq 0$. It follows that $M_{t_1} \neq \phi$ and $M_s \subseteq M_t$, $0 \leq s \leq t$.

Claim If $t^* \in R_+^1$ and $M_{t^*} \notin \{\phi, N\}$, then $M_{t^*} \neq M_{t^*+r}$.

If the claim is not true, then $M_t = M_{t^*}$, $t \in [t^*, t^* + r]$. Thus, it follows from (A12) that there exists $i \in N \setminus M_{t^*+r}$ such that $f_i(x_{t^*+r}(\varphi)) > 0$. Hence, from (1.1), we obtain

$$x_i{}'(t^* + r, \varphi) = f_i(x_{t^*+r}(\varphi)) > 0.$$

Hence, there exists $\varepsilon > 0$ such that

$$\frac{\mathrm{d}(x_i(t, \varphi))}{\mathrm{d}t} > 0 \quad \text{for all } t \in [t^* + r - \varepsilon, t^* + r].$$

Since $x_t(\varphi) \geq 0$ for all $t \geq 0$, we have $x_i(t^*+r, \varphi) > 0$. So, it follows $i \in M_{t^*+r}$, which yields a contradiction. This completes the proof of the claim.

Now, we will show that $M_{t_1+(n-1)r} = N$. Otherwise, by the above claim, we have

$$\phi \neq M_{t_1} \subseteq M_{t_1+r} \subseteq \cdots \subseteq M_{t_1+(n-1)r} \subseteq M_{t_1+nr},$$

and $M_{t_1+ir} \neq M_{t_1+(i-1)r}$, $i \in N$. But this contradicts $M_t \subseteq N$. This completes the proof. □

Theorem 2.2. *Let* (A11), (A12) *and* (A3)–(A5) *hold. Then x^* is a globally stable equilibrium in the set of* $\{\varphi \in C_+ : \varphi(0) > 0\}$.

Proof. Suppose $\varphi \in C_+$ with $\varphi(0) > 0$. Then by Lemma 2.1, we have $x_t(\varphi) \in IntC_+$ for all $t \geq (n+2)r$. Define $V : C_+ \to C_+$ such that $V(\varphi) = \|\varphi - x^*\| \triangleq \sup\{\|\varphi_i(\theta) - x_i^*\| : i \in N \text{ and } \theta \in [-r_i, 0]\}$. Using a similar argument as in Theorem 2.1, we may deduce that the following statements are true:

(i) If there exists $T_0 \geq 0$ such that $x_{T_0}(\varphi) = x^*$, then $x_t(\varphi) = x^*$ for all $t \geq T_0$;
(ii) If $x_t(\varphi) \neq x^*$ for all $t \in R_+^1$, then $V(x_t(\varphi))$ is strictly decreasing on $[(n+2)r, \ +\infty)$;
(iii) $V(x_t(\varphi))$ is non-increasing on $[(n+2)r, \ +\infty)$.

Consequently, Theorem 2.2 follows from Proposition 2.1. This completes the proof of Theorem 2.2. □

Next, consider the following systems of delay different equations:

$$x_i'(t) = L_i x_t + x_i(t) g_i(x_t), \tag{2.2}$$

where $L \triangleq (L_1, L_2, \ldots, L_n) : C_+ \to R^n$ is a bounded linear operator and $g \triangleq (g_1, g_2, \ldots, g_n) : C_+ \to R^n$.

Here, we give the following assumptions:

(C1) If $\varphi \in C_+$ and $\varphi_i(0) = 0$ for some i then $L_i(\varphi) \geq 0$;
(C2) Denote $f(\varphi) = L\varphi + diag\varphi(0)g(\varphi)$ for all $\varphi \in C_+$. There exists $x^* \in IntR_+^n$ such that $f(x^*) = 0$ and f satisfies assumptions (A4) and (A5);
(C3) f satisfies (A12).

In view of (C1), it follows from Proposition 1.2 of [10] that system (2.2) satisfies condition (A1). Moreover, together with (A12), we can show that Lemma 2.1 also holds. Hence, by using a similar argument as in the proof of Theorem 2.2, we obtain the following result.

Theorem 2.3. *Let* (C1)–(C3) *hold. Then x^* is a global stability equilibrium in the set of* $\{\varphi \in C_+ : \varphi(0) > 0\}$.

3. Applications

In this section, we give several examples to illustrate the applications of main results in Section 2.

Example 3.1. Consider the following Nicholson's Blowflies models with the multi-patches:

$$x_i'(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \beta x_i(t - r_i) e^{-x_i(t-r_i)} - d x_i(t), \quad i = 1, 2, \ldots, n, \tag{3.1}$$

where $\beta > 0$, $A = (a_{ij})_{n\times n}$ is a cooperative and irreducible matrix and $(r_1, r_2, \ldots, r_n) \in IntR_+^n$. In what follows, we always assume that $\frac{\beta}{d} \in [e, e^2]$ and $\sum_{j\neq i} a_{ij} = -a_{ii}$ for all i.

Lemma 3.1 ([16, Lemma 2.3]). *Let $\bar{\beta} = \frac{\beta}{d}$. If $a \geq 0$ and $b \geq 0$, then we have the following results:*

(i) *If $a - \ln\bar{\beta} \geq |b - \ln\bar{\beta}|$, then $-a + \bar{\beta} b e^{-b} \leq 0$. Moreover, $-a + \bar{\beta} b e^{-b} = 0$ if and only if $a = b = \ln\bar{\beta}$;*
(ii) *If $\ln\bar{\beta} - a \geq |b - \ln\bar{\beta}|$, then $-a + \bar{\beta} b e^{-b} \geq 0$. Moreover, $-a + \bar{\beta} b e^{-b} = 0$ if and only if either $a = b = \ln\bar{\beta}$ or $a = b = 0$.*

Lemma 3.2. *If $\varphi \in C_+ \setminus \{0\}$, then there exists $t_0 \in [0, r]$ such that $x_i(t_0, \varphi) > 0$ for some i.*

Proof. Otherwise, $x(t, \varphi) = 0$ for all $t \in [0, r]$. From (3.1), we have $x_i(t - r_i, \varphi) e^{-x_i(t-r_i,\varphi)} = 0$ for all $t \in [0, r_i]$ and $i \in N$, and hence $\varphi = 0$, a contradiction. This completes the proof. □

In Example 3.1, let $x^* = (\ln\frac{\beta}{d}, \ln\frac{\beta}{d}, \ldots, \ln\frac{\beta}{d})$, $D = diag\{d, d, \ldots, d\}$ and $f(\varphi) = A\varphi(0) - D\varphi(0) + \beta h(\varphi)$, where $h(\varphi) = diag\{\varphi_1(-r_1)e^{\varphi_1(-r_1)}, \varphi_2(-r_2)e^{\varphi_2(-r_2)}, \ldots, \varphi_n(-r_n)e^{\varphi_n(-r_n)}\}$.

Theorem 3.1. *For system (3.1), if $\varphi \in C_+ \setminus \{0\}$, then $\lim_{t\to\infty} x(t, \varphi) = (\ln\frac{\beta}{d}, \ln\frac{\beta}{d}, \ldots, \ln\frac{\beta}{d})$.*

Proof. Since $A = (a_{ij})_{n\times n}$ is a cooperative and irreducible matrix, by Lemmas 3.1 and 3.2, assumptions (A11), (A12) and (A3)–(A5) hold. Consequently, Theorem 3.1 follows from Theorem 2.2. □

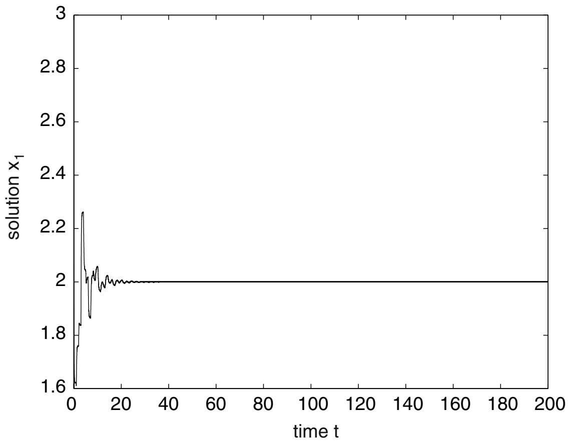

Fig. 1. Numerical solution $x_1(t)$ of system (3.2) for $(\phi_1(s), \phi_2(s)) = (3, 1)$.

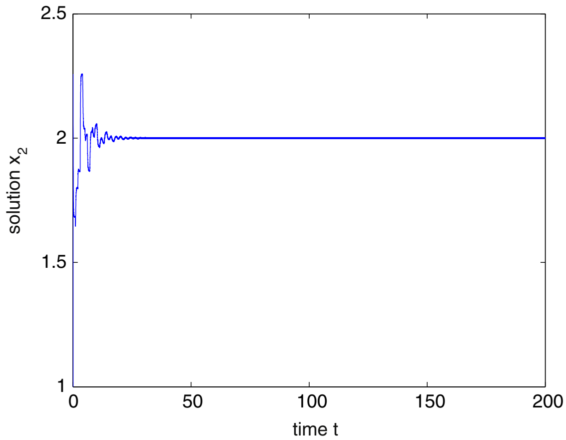

Fig. 2. Numerical solution $x_2(t)$ of system (3.2) for $(\phi_1(s), \phi_2(s)) = (3, 1)$.

Remark 3.1. To illustrate the effectiveness of Theorem 3.1, we consider Example 3.1 with $n = 2$, and

$$\begin{cases} x_1'(t) = (-100x_1(t) + 100x_2(t)) - 12x_1(t) + 3e^2x_1(t-1)e^{-x_1(t-1)} \\ \quad + 2e^2x_1(t-2)e^{-x_1(t-2)} + 7e^2x_1(t-3)e^{-x_1(t-3)}, \\ x_2'(t) = (-200x_2(t) + 200x_1(t)) - 12x_2(t) + 3e^2x_2(t-3)e^{-x_2(t-3)} \\ \quad + 2e^2x_2(t-2)e^{-x_2(t-2)} + 7e^2x_2(t-1)e^{-x_2(t-1)}. \end{cases} \tag{3.2}$$

Then,

$$r = r_1 = r_2 = 3, \quad \beta = 12e^2 > 0, \quad d = 12, \quad \frac{\beta}{d} = e^2 \in [e, e^2].$$

By Theorem 3.1, for system (3.2), if $\varphi \in \{\varphi \in C_+ : \varphi(0) > 0\}$, then $\lim_{t\to\infty} x(t; t_0, \varphi) = (2, 2)$. This fact is verified by the numerical simulation in Figs. 1 and 2.

Example 3.2. Consider the following delay differential systems:

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + k_ix_i(t)\left[\alpha_i - b_{i0}x_i(t) - \sum_{j=1}^{m} b_{ij}x_i(t-r_j)\right], \quad i = 1, 2, \dots, n. \tag{3.3}$$

where $k_i > 0$, $\alpha_i > 0$ and $b_{ij} \le 0, j = 1, 2, \dots, m$. $A = (a_{ij})_{n\times n}$ is a cooperative and irreducible matrix and $(r_1, r_2, \dots, r_n) \in IntR^n_+$.

In this example, denote $b_i = \sum_{j=0}^{m} b_{ij} > 0$, and $\alpha_i^* = \frac{\alpha_i}{b_i}$, $x^* = (\alpha_1^*, \alpha_2^*, \dots, \alpha_n^*)$. Also, denote $f(\varphi) = (f_1(\varphi), f_2(\varphi), \dots, f_n(\varphi))$ where $f_i(\varphi) = \sum_{j=1}^{n} a_{ij}\varphi_j(0) + k_i\varphi_i(0)[\alpha_i - b_{i0}\varphi_i(0) - \sum_{j=1}^{m} b_{ij}\varphi_i(-r_j)]$ for $\varphi \in C_+$.

Lemma 3.3. *If $\alpha_i^* = \alpha_1^*$ and $\sum_{j\neq i} a_{ij} = -a_{ii}$ for all $i \in N$, then f satisfies assumptions* (A3)–(A5) *of Section* 2.

Proof. Obviously, $x^* \in Int(R^n_+)$ and $f(x^*) = 0$, that is, (A3) follows. Let $\varphi_i(0) - x^*_i \geq \|\varphi - x^*\|$ for some i and $\varphi \in C_+ \setminus \{x^*\}$. Then we have $\varphi_i(0) > x^*_i$ and

$$\begin{aligned}
f_i(\varphi) &= \sum_{j=1}^{n} a_{ij}\varphi_j(0) + k_i\varphi_i(0)\left[\alpha_i - b_{i0}\varphi_i(0) - \sum_{j=1}^{m} b_{ij}\varphi_i(-r_j)\right] \\
&\leq \sum_{j=1}^{n} a_{ij}\varphi_i(0) + k_i\varphi_i(0)\left[\alpha_i - b_{i0}\varphi_i(0) - \sum_{j=1}^{m} b_{ij}\varphi_i(-r_j)\right] \\
&\leq \sum_{j=1}^{n} a_{ij}\varphi_i(0) + k_i\varphi_i(0)\left[\alpha_i - b_{i0}\varphi_i(0) - \sum_{j=1}^{m} b_{ij}\varphi_i(0)\right] \\
&\leq k_i\varphi_i(0)\left[\alpha_i - b_{i0}\varphi_i(0) - \sum_{j=1}^{m} b_{ij}\varphi_i(0)\right] \\
&\leq -k_i\varphi_i(0)\left(b_{i0} + \sum_{j=1}^{m} b_{ij}\right)(\varphi_i(0) - \alpha^*_i) \\
&< 0.
\end{aligned}$$

Thus, (A4) holds. By a similar argument as above, we can deduce that (A5) follows. This completes the proof. □

In what follows, we always assume that $\alpha^*_i = \alpha^*_1$ and $\sum_{j\neq i} a_{ij} = -a_{ii}$ for all $i \in N$.

Lemma 3.4. *If $\varphi \in C_+$, then $x_t(\varphi)$ exists and is unique on R^1_+.*

Proof. Suppose $\varphi \in C_+$, Obviously, from Proposition 1.2 of [4], we obtain that $x_t(\varphi) \in C_+$ is unique for $t \in [0, \eta)$, where $[0, \eta)$ is the maximal right-interval of the existence of $x(t, \varphi)$. We next show that $\eta = \infty$. Denote $M_t = \sup\{|x_i(t + \theta, \varphi) - x^*_i| : i \in N \text{ and } \theta \in [-r_i, 0]\}$ for all $t \in [0, \eta)$. We now claim that M_t is non-increasing on $[0, \eta)$. Otherwise, without loss of generality, we may assume that there exist $i \in N$ and $t^* \geq 0$ such that $x_i(t^*, \varphi) - \alpha^*_i > \sup\{|x_i(t, \varphi) - x^*_i| : i \in N \text{ and } t \in [t^* - r_i, t^*)\}$ and hence $x'_i(t^*, \varphi) \geq 0$. On the other hand, it follows from (3.3) that

$$\begin{aligned}
x'_i(t^*, \varphi) &= \sum_{j=1}^{n} a_{ij}x_j(t^*, \varphi) + k_ix_i(t^*, \varphi)\left[\alpha_i - b_{i0}x_i(t^*, \varphi) - \sum_{j=1}^{m} b_{ij}x_i(t^* - r_j, \varphi)\right] \\
&\leq k_ix_i(t^*, \varphi)\left[\alpha_i - b_{i0}x_i(t^*, \varphi) - \sum_{j=1}^{m} b_{ij}x_i(t^* - r_j, \varphi)\right] \\
&\leq -k_ix_i(t^*, \varphi)\left(b_{i0} + \sum_{j=1}^{m} b_{ij}\right)(x_i(t^*, \varphi) - \alpha^*_i) \\
&< 0.
\end{aligned}$$

This yields a contradiction and hence Claim follows. Consequently, $M_t \leq M_0$ for all $t \in [0, \eta)$ and thus $x_i(t, \varphi) \leq M_0 + \alpha^*$ for all $t \in [0, \eta)$. Therefore, by Theorem 3.1 in [17], $\eta = \infty$. This completes the proof. □

Theorem 3.2. *For system (3.3), if $\varphi \in C_+ \setminus \{0\}$, then* $\lim_{t\to\infty} x(t, \varphi) = (\alpha^*_1, \alpha^*_2, \ldots, \alpha^*_n)$.

Proof. Since $A = (a_{ij})_{n\times n}$ is a cooperative and irreducible matrix, by Lemmas 3.3 and 3.4, assumptions (C1)–(C3) hold. Consequently, Theorem 3.2 follows from Theorem 2.3. □

4. Conclusion

In this paper, a class of systems of delay differential equations have been studied. Under some appropriate conditions, a "proper" Lyapunov functional is constructed to study the global stability of positive equilibrium. Without considering the derivatives of Lyapunov functional, several criteria of the global stability of positive equilibrium have been established on the nonnegative function space. Our results can be applied for some practical problems concerning population dynamics and ecology problems. These obtained results are new and they complement previously known results. Moreover, two examples are given to illustrate the effectiveness of our new results. In the real world, the delays in population dynamics and ecology problems are usually time-varying. Whether or not our results and method in this paper are available for this case, it is an interesting problem and we leave it as our work in the future.

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